

BLACK HOLES FROM BLUE SPECTRA

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ABSTRACT

Blue primordial power spectra with a spectral index $n > 1$ can lead to a significant production of primordial black holes in the very early Universe. The evaporation of these objects leads to a number of observational consequences and a model independent upper limit of $n \approx 1.4$. In some cases this limit is strengthened to $n = 1.3$. Such limits may be employed to define the boundary to the region of parameter space consistent with generalized inflationary predictions. [To appear in Proceedings of the CASE WESTERN CMB WORKSHOP, April 22-24 1994. Figures available on request from J.H.Gilbert@qmw.ac.uk]

1. Introduction: Towards an Observational Test of Inflation

The positive detection of anisotropic structure in the temperature distribution of the Cosmic Microwave Background (CMB) radiation has opened up the possibility that the predictions of the inflationary scenario may be testable within the near future^{1,2}. During inflation the scale factor grows exponentially, whilst the Hubble radius $H^{-1} \sim 10^{-23}\text{cm}$ remains almost constant. Consequently the physical wavelength of a quantum fluctuation in the scalar or graviton field soon exceeds H^{-1} and its amplitude then becomes ‘frozen’. Once inflation has ended, however, H^{-1} increases faster than the scale factor, so the fluctuations eventually reenter the Hubble radius during the radiation- or matter-dominated eras. The fluctuations that exit around 60 e-foldings or so before reheating reenter with physical wavelengths in the astrophysically interesting range 1 Mpc - 10^4 Mpc. Fluctuations in the graviton degrees of freedom result in a stochastic background of primordial gravitational waves that have an amplitude at reentry given by $\delta_{\text{GW}} \approx V/m_{\text{Pl}}^4$, where m_{Pl} is the Planck mass. Fluctuations in the inflaton field provide the seeds for galaxy formation via gravitational instability and their amplitude at reentry is $\delta \approx V^{3/2}/(m_{\text{Pl}}^3|V'|)$. Both scalar and tensor fluctuations lead to CMB anisotropies.

Since H decreases as the inflaton field rolls down the potential, the amplitude of the scalar and tensor fluctuations is scale-dependent. This variation is most conveniently parametrized in terms of the spectral indices, n_T and n , defined by $\delta_{\text{GW}}(M) \propto M^{-n_T/6}$ and $\delta(M) \propto M^{(1-n)/6}$, where M is the mass scale associated with the Hubble radius at the epoch of reentry. In general these spectral indices

are themselves functions of scale. However, scales relevant to large-scale structure and CMB experiments probe only 9 e-foldings or so of the inflationary expansion, and since the scalar field must be rolling slowly for inflation to occur in the first place, these scales correspond to a very small portion of the inflationary potential. Hence, it is reasonable to suppose that the spectral indices are indeed constant over the scales of interest³.

When comparing theory with experiment, it is conventional to expand the CMB temperature fluctuations on the sky in terms of spherical harmonics. The angular correlation function predicted from theory is then given by an average over all observer positions:

$$\langle \delta T(\theta) \delta T(0) \rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l P_l(\cos \theta), \quad (1)$$

where the P_l 's are Legendre polynomials and a given multipole l corresponds to an angular scale $\theta/1^\circ \approx 60/l$. The C_l 's corresponding to the scalar ($C_{l,S}$) and tensor ($C_{l,T}$) fluctuations are determined once the precise functional forms of the spectra have been specified. In the limit that $|n_T|$ and $|n-1|$ are constant and small, the ratio of the $l=2$ multipoles is uniquely determined by the tensor spectral index:

$$R \equiv \frac{C_{2,T}}{C_{2,S}} \approx -7n_T. \quad (2)$$

This *consistency equation* is a fairly generic prediction of inflation⁴. The quantities in this expression are measurable, at least in principle, and it forms the basis for an observational test of the scenario. If no reionization occurs, experiments on scales $\theta \geq 2^\circ$ ($l \leq 30$) provide a measure of the sum of the scalar and tensor contributions: $C_l = C_{l,S} + C_{l,T}$. Since the gravitational waves do not produce a measurable contribution to the CMB anisotropy for $\theta \leq 2^\circ$, one might hope to determine $C_{2,T}$ and $C_{2,S}$ separately from a combination of small and large angle CMB experiments. A test of inflation would then require a separate determination of the tensor spectral index. At present it seems that this requires a direct detection of the gravitational wave spectrum. Recent calculations⁵ suggest that the maximum present-day contribution per octave of the gravitational waves is $\Omega_{\text{GW}} h^2 \leq 7 \times 10^{-15}$. This is too weak to be detectable by the Laser Interferometer Gravity-Wave Observatories, although the proposed beam-in-space experiment has a peak sensitivity of $\Omega_{\text{GW}} \approx 10^{-16}$ at 10^{-4} Hz and may be able to detect such a background.

2. Constraining the Scalar Spectral Index with PBHs

The scalar spectral index is much easier to measure and we would like to have an expression equivalent to the consistency equation that relates the C_2 's to n . In general, the relationship between n_T and n is model dependent, but for the special case of an exponential potential, we have $n-1 = n_T$. Hence, in the parameter space (R, n) , this model corresponds to the line $R = 7(1-n)$. It is important to note, however, that other families of potentials will lead to different trajectories in

this space. For example, any potential of the form $V = V_0[1 \pm 2\pi|n - 1|\phi^2/m_{\text{Pl}}^2]$ leads to a measurable deviation of n from unity, whilst predicting that $R \approx 0$.⁶ One could imagine taking all the current candidates for the inflationary potential and predicting their separate trajectories in this parameter space². A superposition of these paths would then define a target that represented a generalized prediction of inflation in some sense. The problem with this approach, however, would be in deciding which models should be included in the target. Whilst a given model may appear to be natural to one person, someone else may deem it unnatural.

It seems to us that this subjective element in the testing procedure must be eliminated if such an approach is to be developed further. To accomplish this it is necessary to search for model-independent constraints on the parameters R and n . Recently an upper limit on the spectral index was derived from considering the formation and subsequent evaporation of primordial black holes (PBHs)⁷. This limit is independent of any gravitational wave contribution to the CMB anisotropy and therefore defines a boundary to the observational target of inflationary predictions in (R, n) space.

The idea behind the argument is rather simple. During inflation the first scales to leave the Hubble radius are the last to come back in and this implies that the very last fluctuation to leave is the first to return. In the simplest case, the fluctuation on this scale will be spherically symmetric and Gaussian distributed with an rms amplitude given by $\delta(t_e)$, where $t_e \sim H^{-1}$ is the time when inflation ends. In some regions of the post-inflationary Universe, the fluctuation will be sufficiently large that the collapse of a local region into a black hole will become inevitable. The higher the rms amplitude the higher the fraction of the Universe forming PBHs. The observational consequences of the evaporation of these black holes then leads to upper limits on the number that may form and hence on the magnitude of $\delta(t_e)$. An upper limit on n is therefore derived by normalizing the power spectrum on the quadrupole scale, $M_Q \sim 10^{57}\text{g}$, and assuming that the spectral index is constant.

PBHs are never produced in sufficient numbers to be interesting if $n < 1$, but they could be if the spectrum is ‘blue’ with $n > 1$. The precise form of the constraints depends crucially on how the Universe is reheated, however, and we now proceed to discuss the separate cases of efficient and inefficient reheating.

2.1. Efficient Reheating

When an overdense region with equation of state $p = \gamma\rho$ stops expanding, it must have a size greater than $\sqrt{\gamma}$ times the horizon size in order to collapse against the pressure and this requires that $\delta(t_e) > \gamma$. It follows that the probability of a region of mass M forming a PBH is⁸

$$\beta_0(M) \approx \delta(M) \exp\left(-\frac{\gamma^2}{2\delta^2(M)}\right). \quad (3)$$

Because of the exponential expansion PBHs that form before or during inflation have no observational consequences. Furthermore, if the reheating process is very

efficient, the false vacuum energy is rapidly converted into relativistic particles with a reheating temperature $T_{\text{RH}}/T_{\text{Pl}} = (t_e/t_{\text{Pl}})^{-1/2}$. The mass of a PBH forming at this time is $M_{\text{RH}}/m_{\text{Pl}} \approx t_{\text{RH}}/t_{\text{Pl}}$ and once it has formed, a PBH of this mass will evaporate at a time $t_{\text{evap}} \approx (M_{\text{RH}}/m_{\text{Pl}})^3 t_{\text{Pl}}$. Eq. (3) then implies that for a blue spectrum, $\beta_0(M)$ decreases exponentially for $M > M_{\text{RH}}$, so we may regard the PBH mass spectrum as effectively being a δ -function at M_{RH} .

The constraints on $\beta_0(M)$ in the range $10^{10}\text{g} \leq M \leq 10^{17}\text{g}$ were recently summarized⁹. In particular, PBHs with an initial mass $\sim 10^{15}\text{g}$ would evaporate at the present epoch and may contribute appreciably to the observed gamma-ray and cosmic-ray spectra at 100 MeV. On the other hand, 10^{10}g PBHs have a lifetime ~ 1 sec and, if produced in sufficient numbers, their evaporations would lead to the photodissociation of deuterium immediately after the nucleosynthesis era. PBHs of mass slightly below 10^{10}g could alter the baryon-to-photon ratio just prior to nucleosynthesis. In our paper⁷ we consider the constraints on $\beta_0(M)$ below 10^{10}g . In this region there is a potentially stronger constraint on the spectral index if evaporating PBHs leave stable Planck mass relics¹⁰. Although the formation of such objects has not been proved conclusively, it would be surprising if quantum gravity effects did not become important once the PBH had evaporated down to the Planck mass and various arguments have been developed in the literature suggesting the formation of such objects is likely⁷.

To derive the observational constraint from PBH relics one proceeds as follows: a lower limit on M_{RH} is derived by assuming that the observed quadrupole anisotropy is due entirely to gravitational waves. This implies⁵ that the expansion rate of the Universe during the last 60 e-foldings of inflation cannot exceed $3 \times 10^{-5} m_{\text{Pl}}$ and leads to an upper limit on the reheat temperature of $\sim 10^{16}$ GeV. This corresponds to a minimum mass $\sim 1\text{g}$. The observational constraint from the relics derives from the fact that they cannot have more than the critical density at the present epoch, i.e. $\Omega_{\text{rel}} < 1$. The precise form of the constraint depends on whether the evaporating PBHs dominate the energy density of the Universe before they evaporate. Since the ratio of PBH density to radiation density increases as $t^{1/2}$, the PBHs do not dominate at evaporation if $\beta_0 < (M_{\text{RH}}/m_{\text{Pl}})^{-1}$. If this condition is satisfied, the constraint that Ω_{rel} does not exceed unity becomes

$$\beta_0(M) < 10^{-27} \left(\frac{M}{m_{\text{Pl}}} \right)^{3/2}. \quad (4)$$

If, on the other hand, the PBHs dominate the density at evaporation, most of the background photons derive from the PBHs and the constraint becomes $M > 10^6\text{g}$. In other words, only relics formed from PBHs smaller than 10^6g can contribute significantly to the current energy density and above this critical mass the entropy constraint takes over.

We have completed a detailed analysis and derived the constraints on $\delta(M_{\text{RH}})$ for all mass scales above 1g . These results are illustrated in Figure 1. It is seen that the strongest limit on the spectral index derives from the relic constraint and

by combining Eqs. (3) and (4) one finds that

$$\delta(M_{\text{RH}}) < 0.13 \left[17 - \log_{10} \left(\frac{M_{\text{RH}}}{m_{\text{Pl}}} \right) \right]^{-1/2}. \quad (5)$$

If we normalize on the quadrupole scale, M_Q , the rms amplitude on a smaller scale M is $\delta(M) = \delta_Q (M/M_Q)^{(1-n)/6}$, where $\delta_Q \approx 3.8 \times 10^{-6}$. Hence, we conclude from Eq. (5) that the limit on the spectral index is $n \leq 1.4$ for $M_{\text{RH}} \approx 1\text{g}$, corresponding to a reheat temperature $\sim 10^{16}\text{GeV}$, and $n \leq 1.5$ for $M_{\text{RH}} \approx 10^6\text{g}$, corresponding to $T_{\text{RH}} \sim 10^{14}\text{GeV}$. The constraints associated with higher masses (i.e. lower reheat temperatures) are calculated by a similar procedure and are summarized in Figure 2.

Fig. 1. The constraints on the rms amplitude of the scalar fluctuation spectrum immediately after inflation if the equation of state is radiation-like ($\gamma = 1/3$). The origin of the constraints above 10^{10}g is summarized by Carr and Lidsey⁹. If PBHs do not leave behind stable relics after evaporation, the strongest upper bound on the spectral index is given by the dashed line which joins the COBE/DMR point and the deuterium constraint at 10^{10}g . This limit applies for reheat temperatures $\sim 10^9\text{GeV}$. If relics are formed, the limit is strengthened at higher reheat temperatures as indicated.

2.2. Inefficient Reheating

Thus far we have assumed that the reheating of the Universe to relativistic particles occurs on a timescale much less than H^{-1} . However, when inflation ends by means of a second-order phase transition, the scalar field undergoes coherent oscillations in the potential minimum from the time $t_1 \sim H^{-1}$ until it decays at a time $t_2 \sim \Gamma^{-1}$, where Γ is its decay width. During this interval, the Universe is effectively dominated by a dust-like fluid¹¹ with $\gamma = 0$, and Eq. (3) does not apply. PBHs will still form in this case, but the fraction of the Universe going into PBHs is now determined by the probability that regions are sufficiently spherically symmetric to collapse within their Schwarzschild radius. This fraction is given by¹²

$$\beta(M) \approx 2 \times 10^{-2} [\delta(M)]^{13/2} \quad (6)$$

and the observational constraints on the probability of PBH formation are altered because they now have an extended mass spectrum. The range over which this spectrum applies is defined by $M_1 \leq M \leq M_{\max}$, where M_1 is the horizon mass immediately after inflation and M_{\max} is the mass of a configuration that just detaches itself from the universal expansion at t_2 . It is determined implicitly by¹²

$$M_{\max} = [\delta(M_{\max})]^{3/2} \left(\frac{t_2}{t_{\text{Pl}}} \right) m_{\text{Pl}}. \quad (7)$$

The constraints on $\beta(M)$ are related to the associated constraints on $\beta_0(M)$ via the relation

$$\beta(M) = \beta_0(M) \left(\frac{t_2}{t_{\text{Pl}}} \right)^{1/2} \left(\frac{M}{m_{\text{Pl}}} \right)^{-1/2}. \quad (8)$$

This is a useful expression because it implies that the limits on PBH formation during the dust phase can be calculated directly from the constraints which apply if there is no dust phase. We first assume t_1 is fixed and vary the epoch $t_2 = \Gamma^{-1}$ at which the dust era ends. For a given value of t_2 , the mass range of PBHs forming during the dust phase goes from M_1 to the mass given by Eq. (7) and for each value of M the constraint on $\beta(M)$ is given by Eq. (8). The corresponding limit on $\delta(M)$ follows from Eq. (6) and the limit on n is derived by normalizing on the quadrupole scale as before. The reheat temperature is $T_{\text{RH}} \approx (\Gamma t_{\text{Pl}})^{1/2} m_{\text{Pl}}$ and an upper limit on Γt_{Pl} follows from the requirement that baryogenesis must proceed after reheating. It is generally accepted that the lowest temperature for which the observed baryon asymmetry may be generated is the electroweak scale, $\sim 10^3 \text{ GeV}$, corresponding to $\Gamma t_{\text{Pl}} \sim 10^{-30}$.

The new constraints on the spectral index are also shown in Figure 2. For a relatively short dust phase, only the relic limit will be altered, since PBHs above 10^{10} g will not form during the dust phase. For lower reheat temperatures, however, more massive PBHs form and provide the strongest constraint. The deuterium constraint applies for $10^{-17} \geq \Gamma t_{\text{Pl}} \geq 10^{-23}$ and the gamma-ray limit for $10^{-23} \geq$

$\Gamma t_{\text{Pl}} \geq 10^{-30}$. From this figure we arrive at an upper limit of $n = 1.4$ if PBHs form relics and there is a dust-phase after inflation.

Fig. 2. Illustrating the constraints on the spectral index arising from the overproduction of primordial black holes, the shaded area being excluded. The lower line applies if there is a dust phase immediately after inflation, in which case the ordinate is $\log_{10} \Gamma t_{\text{Pl}}$, the upper line if there is no dust phase in which case it is $\log_{10}(t_{\text{Pl}}/t_e)$. The constraints depend on the reheat temperature $T_{\text{RH}} \approx 10^{18}(\Gamma t_{\text{Pl}})^{1/2} \text{GeV}$, where Γ is the decay width of the scalar field that decays into relativistic particles. The n -independent upper and lower limits on the decay width arise from assuming the COBE/DMR detection is due entirely to gravitational waves (shaded line) and from requiring that baryogenesis can only proceed above the electroweak scale (dashed line). The dotted horizontal line indicates the CMB distortion limit of Hu et al ¹³. For reheat temperatures above $T_{\text{RH}} \approx 10^{9.5} \text{GeV}$ the most important constraint arises from the requirement that any Planck mass relics left over from the final stages of PBH evaporation should have less than the critical density at the present epoch. For lower reheat temperatures, more massive PBHs may form and the strongest constraints then arise from the photodissociation of deuterium by evaporating 10^{10}g PBHs, from the observed gamma-ray background in the energy range $0.1 - 1 \text{ GeV}$, or from the distortions of the CMB.

3. Conclusion

One criticism of these limits is that they assume the spectral index is constant over the full range of scales. To answer this point we note that the general inflation-

any potential leading to spectra with *constant* spectral index $n > 1$ is a combination of trigonometric functions with a Taylor expansion of the form⁹

$$V = V_0 \left[1 + 2\pi(n-1) \frac{\phi^2}{m_{\text{Pl}}^2} \right]. \quad (9)$$

Any potential that leads to $n > 1$ will have a Taylor expansion of this form. It follows that, as the field rolls towards the minimum, the approximation of Eq. (9) to the general trigonometric potential becomes more accurate and so the variations in the spectral index become smaller. Consequently, one need only show that the spectral index is effectively constant over the scales corresponding to large-scale structure (53-60 e-foldings from the end of inflation). It is straightforward to show that this is the case⁷.

We conclude that the formation of PBHs from quantum vacuum fluctuations immediately after inflation constrains the scalar spectral index to be less than 1.4. It is important to emphasize that, because our limit spans the large range of scales from $\sim 1\text{g}$ to $\sim 10^{57}\text{g}$, (for comparison, all large-scale structure measurements span only 10 decades of scale), it is essentially independent of the errors that arise in the COBE normalization from possible tensor contributions, cosmic variance and the effect of the Doppler peak on the low multipole anisotropies. Moreover, since we have normalized on COBE scales, the limit is also independent of the precise form of dark matter and hence the bias parameter. In effect, once the COBE normalization is specified, the limit is independent of the cosmological model, although it does assume that $\Omega_0 = 1$, as predicted by most inflationary models. Therefore this limit can be employed to define a boundary to the target of inflationary predictions in (R, n) space.

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